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BAYES ESTIMATOR OF GENERALIZED-EXPONENTIAL PARAMETERS UNDER GENERAL ENTROPYOSS FUNCTION USING LINDLEY'S APPROXIMATION

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ABSTRACT

In this paper, we have obtained the Bayes Estimator of scale and shape parameter of Generalized-Exponential using Lindley's approximation (L-approximation) under GENERAL ENTROPY loss functions. The proposed estimators have been compared with the corresponding MLE for their risks based on simulated samples from the Generalized-Exponential distribution.

1 Introduction

Exponential distribution is the most exploited distribution for life data analysis, but its suitability is restricted to constant hazard rate. For situations where the failure rate is monotonically increasing or decreasing, two-parameter Weibull and Gamma are the most popular distributions used for analyzing any lifetime data. Both distributions have increasing / decreasing hazard rates depending on the shape parameter. However, one of the major disadvantages of the gamma distribution is that its distribution and survival functions cannot be expressed in a closed form if the shape parameter is not an integer. Moreover, even for integer shape parameter, there are terms involving the incomplete gamma function. Thus, one needs numerical integration to obtain distribution function, survival function, or hazard function. Perhaps, this makes the gamma distribution unpopular compared to a Weibull distribution, which has a nice closed form for the hazard and survival functions. On the other hand, the Weibull distribution has

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its own disadvantages. For example, Bain and Engelhardt (1991) have pointed out that the maximum likelihood estimators of a Weibull distribution might not behave properly for all parameter ranges.

Recently a new distribution, called Generalized-Exponential distribution, has been introduced. This distribution can be used quite effectively in situations where a skewed distribution is needed. Gupta and Kundu (1999, 2002) and Raqab and Ahsanullah (2001) have investigated several properties of the two parameter generalized exponential distribution.

The two-parameter Generalized-Exponential has a distribution function of the form

$$F_{GE}\left(x \mid \alpha, \lambda\right) = \left(1 - e^{-\lambda x}\right)^{\alpha} \ ; \ \alpha \ , \ \lambda \ > \ 0 \tag{1}$$

and hazard function given by

$$h\left(x \mid \alpha, \lambda\right) = \frac{\alpha\lambda\left(1 - e^{-\lambda x}\right)^{\alpha - 1} e^{-\lambda x}}{1 - \left(1 - e^{-\lambda x}\right)^{\alpha}} \tag{2}$$

Here α is the shape parameter, and λ is the scale parameter. The two-parameter Generalized-Exponential has increasing / decreasing failure rates depending on the shape parameter. For any λ, the hazard function is increasing if $\alpha > 1$, decreasing if $\alpha < 1$, and constant if $\alpha = 1$. Gupta and Kundu (1999) observed that because of the simple structure of the distribution and survival functions, the two-parameter Generalized-Exponential can be used quite effectively in analyzing many lifetime data, particularly in place of two-parameter gamma and Weibull distributions.

The estimation of parameters of the Generalized-Exponential distribution has been attempted by Gupta and Kundu (1999), but that work was only concerned with the maximum likelihood estimator or a Bayes estimator under a symmetric loss function. It is remarkable that most of the Bayesian inference procedures have been developed with the usual squared-error loss function, which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. Such a restriction may be impractical in most of the situations of practical importance. For example, in the estimation of reliability and failure rate functions, an overestimation is usually much more serious than an underestimation. In this case, the use of symmetrical loss function might be inappropriate. A similar comment has been made by Basu and Ebrahimi (1991). A useful asymmetric loss known as the LINEX loss function (linear-exponential) was introduced by Varian (1975) and has been widely used by several authors, Zellner (1986), Calabria and Pulcini (1996), Soliman (2002), Singh et al. (2005), Ahmadi et al. (2005) and Singh et al. (2008). This function rises approximately exponentially on one side of zero and approximately linearly on the other side. It may also be noted here that the squared-error loss function can be obtained as a particular member of the LINEX loss function for a specific choice of the loss function parameter.

Despite the flexibility and popularity of the LINEX loss function for the location parameter estimation, it appears to be unsuitable for the scale parameter and other quantities (c.f. Basu and Ebrahimi (1991) and Parsian and Sanjari Farsipour (1993)). Keeping these points in mind, Basu and Ebrahimi (1991) defined a modified LINEX loss. A suitable alternative to the modified LINEX loss is the General ENTROPY loss proposed by Calabria and Pulcini (1994a), defined as

$$L_E\left(\bar{\alpha},\alpha\right) \propto \left(\bar{\alpha}/\alpha\right)^c - c\log\left(\bar{\alpha}/\alpha\right) - 1 \tag{3}$$

where $\bar{\alpha}$ is the estimate of parameter α.

This loss is a generalization of the ENTROPY loss used by several authors (see for example, Dey et al. (1987) and Dey and Liu (1992)) where the shape parameter c is taken equal to 1.

The general version (3) allows different shapes of the loss function. It may be noted that when c>0, a positive error causes more serious consequences than a negative error. On the other hand when c<0, a negative error causes more serious consequences than a positive error.

The Bayes estimate $\bar{\alpha}_E$ of α under General ENTROPY loss is given as

$$\bar{\alpha}_E = \left[E_\alpha\left\{\alpha^{-c}\right\}\right]^{-1/c} \tag{4}$$

provided that $E_\alpha\left\{\alpha^{-c}\right\}$ exists and is finite. It can be shown that, when c=1, the Bayes estimate (4) coincides with the Bayes estimate under the weighted squared-error loss function. Similarly, when c= -1 the Bayes estimate (4) coincides with the Bayes estimate under squared error loss function.

This paper considers the Problem of estimating the shape and scale Parameter of Generalized-Exponential distribution using General ENTROPY loss function. It is worthwhile to mention here that posterior distribution of shape and scale parameter for Generalized-Exponential distribution given in (1) involves integral expression in the denominator which can not be reduced in a nice closed form and hence the exact evaluation of posterior expectations for obtaining Bayes estimator for shape and scale parameter will not be possible. We have used Lindley Approximation to obtain Bayes estimators under symmetric and asymmetric loss function.

In the last section, the proposed estimators have been compared with those under quadratic loss function in term of their risk. The comparison is based on

Monte-Carlo study of 2000 simulated sample from Generalized-Exponential distribution.

2. Estimation of parameters

Suppose $x_1,x_2,\ldots,x_n$ is a random sample of size n from the distribution function defined in (2). The likelihood function of λ and α for the samples $x_1,x_2,\ldots,x_n$ is

$$L(\lambda,\alpha \mid x) = \alpha^n \lambda^n \left[\prod_{x=1}^{n}\left(1-e^{-\lambda x_i}\right)\right]^{\alpha-1} \exp\left[-\lambda\sum_{i=1}^{n} x_i\right] : x_i \ge 0 \quad (5)$$

2.1. Maximum Likelihood Estimators of Generalized-Exponential Distributions

The maximum likelihood estimate of parameters of the Generalized-Exponential distribution is obtained by differentiating the log of the likelihood and equating to zero. The two normal equations thus obtained are given below:

$$\frac{n}{\lambda} - \left(\frac{n}{\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)}+1\right)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)} - \sum_{i=1}^{n} x_i = 0 \quad (6)$$

and

$$\hat{\alpha} = \left(\frac{n}{\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)}+1\right) \quad (7)$$

But these normal equations are not solvable. Therefore, the MLE does not exist in a nice closed form. However, the maximum likelihood estimator of two-parameter Generalized-Exponential distribution can be obtained by iterative procedures. We propose here to use bisection method for solving the above-mentioned normal equations.

2.2. Bayes Estimator of Generalized-Exponential Distributions

For Bayesian estimation, we need prior distribution for the parameters α and λ. It may be noted here that when the shape parameter is equal to one, the generalized exponential distribution reduces to exponential distribution. Hence, gamma prior may be taken as the prior distribution for the scale parameter of the Generalized-Exponential distribution. It may be noted here that under the above-mentioned situation, the gamma prior is a conjugate prior. On the other hand, if both the parameters are unknown, a joint conjugate prior for the parameters does not exist. In such a situation, there are a number of ways to choose the priors. We propose the use of piecewise independent priors for both the parameters, namely a non-informative prior for the shape parameter and a natural conjugate prior for the scale parameter (under the assumption that shape parameter is known). Thus the proposed priors for parameters α and λ may be taken as

$$g_1(\alpha) \propto \frac{1}{\alpha}, \quad 0 \le \alpha < \infty, \tag{8}$$

and

$$g_2(\lambda) \propto \frac{m^c \lambda^{c-1} \exp(-m\lambda)}{\Gamma(c)}, \quad \lambda \ge 0; m, c > 0 \tag{9}$$

respectively, to give the joint prior distribution for λ and α as:

$$g(\alpha,\lambda) = \frac{m^c \lambda^{c-1} \exp(-m\lambda)}{\alpha\Gamma(c)}, \lambda \ge 0; m, c > 0 \quad 0 \le \alpha < \infty \tag{10}$$

Substituting $L(\alpha,\lambda \mid x)$ and $g(\alpha,\lambda)$ from (5) and (10) respectively in the following equation

$$P(\alpha,\lambda \mid x) = \frac{L(\alpha,\lambda \mid x).g(\alpha,\lambda)}{\int_0^\infty\int_0^\infty L(\alpha,\lambda \mid x).g(\alpha,\lambda)\, d\lambda\, d\alpha},$$

we get the joint posterior $P(\alpha,\lambda \mid x)$ as

$$P(\alpha,\lambda \mid x) = \frac{\alpha^{n-1}\lambda^{n+c-1}\left[\prod_{x=1}^{n}\left(1-e^{-\lambda xi}\right)\right]^{\alpha-1} \exp\left[-\lambda\left(m+\sum_{i=1}^{n} X_i\right)\right]}{\int_0^\infty\int_0^\infty \alpha^{n-1}\lambda^{n+c-1}\left[\prod_{x=1}^{n}\left(1-e^{-\lambda xi}\right)\right]^{\alpha-1} \exp\left[-\lambda\left(m+\sum_{i=1}^{n} X_i\right)\right] d\lambda d\alpha} \tag{11}$$

Note that the posterior distribution of (α,λ) takes a ratio form, that involves integration in the denominator, which cannot be reduced in nice closed form. Hence, the evaluation of the posterior expectation for obtaining the Bayes estimator of α and λ will be tedious and it will be ratio of unsolvable integrals. Among the various methods suggested to approximate such ratio of integrals, perhaps the simplest one is Lindley's approximation method, which approximates the ratio of the integrals as a whole and produces a single numerical result. Thus, we propose the use of Lindley's (1980) approximation for obtaining the Bayes estimator of α and λ. Many authors have used this approximation for obtaining the Bayes estimators for some lifetime distributions; see among others, Howlader and Hossain (2002) and Jaheen (2005), Singh et al. (2008).

If n is sufficiently large, according to Lindley (1980), any ratio of the integral of the form

$$I(x)=E\left[u(\lambda,\alpha\mid x)\right]=\frac{\int\limits_{(\lambda,\alpha)} u(\lambda,\alpha)e^{L(\lambda,\alpha)+G(\lambda,\alpha)}d(\lambda,\alpha)}{\int\limits_{(\lambda,\alpha)} e^{L(\lambda,\alpha)+G(\lambda,\alpha)}d(\lambda,\alpha)}, \qquad (12)$$

Where $u(\lambda,\alpha)$ = function of λ and α only ;
$L(\lambda,\alpha)$= log of likelihood;
$G(\lambda,\alpha)$= log of joint prior of λ and α .

can be evaluated as

$$I(x)=u(\hat{\lambda},\hat{\alpha})+\frac{1}{2}\Big[(\hat{u}_{\lambda\lambda}+2\hat{u}_{\lambda}\hat{p}_{\lambda})\hat{\sigma}_{\lambda\lambda}+(\hat{u}_{\alpha\lambda}+2\hat{u}_{\alpha}\hat{p}_{\lambda})\hat{\sigma}_{\alpha\lambda}$$

$$+(\hat{u}_{\lambda\alpha}+2\hat{u}_{\lambda}\hat{p}_{\alpha})\hat{\sigma}_{\lambda\alpha}+(\hat{u}_{\alpha\alpha}+2\hat{u}_{\alpha}\hat{p}_{\alpha})\hat{\sigma}_{\alpha\alpha}\Big]$$

$$+\frac{1}{2}\begin{bmatrix}(\hat{u}_{\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{u}_{\alpha}\hat{\sigma}_{\lambda\alpha})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha})+\\(\hat{u}_{\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})\end{bmatrix} \qquad (13)$$

where $\hat{\lambda}$ = M.L.E. of λ ;
$\hat{\alpha}$ = M.L.E. of α.

$$\hat{u}_{\lambda}=\frac{du(\hat{\lambda},\hat{\alpha})}{d\hat{\lambda}};\hat{u}_{\alpha}=\frac{du(\hat{\lambda},\hat{\alpha})}{d\hat{\alpha}};\hat{u}_{\lambda\alpha}=\frac{d^2u(\hat{\lambda},\hat{\alpha})}{d\hat{\lambda}d\hat{\alpha}};$$

$$\hat{u}_{\alpha\lambda}=\frac{d^2u\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}\,d\hat{\lambda}}\hat{u}_{\lambda\lambda}=\frac{d^2u\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}^2};\hat{u}_{\alpha\alpha}=\frac{d^2u\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}^2}$$

$$\hat{L}_{\lambda\lambda\alpha}=\frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}.d\hat{\lambda}.d\hat{\alpha}};\hat{L}_{\lambda\lambda\lambda}=\frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}.d\hat{\lambda}.d\hat{\lambda}};\hat{L}_{\lambda\alpha\lambda}=\frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}.d\hat{\alpha}.d\hat{\lambda}};$$

$$\hat{L}_{\alpha\alpha\lambda}=\frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}.d\hat{\alpha}.d\hat{\lambda}};\hat{L}_{\alpha\lambda\lambda}=\frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}.d\hat{\lambda}.d\hat{\lambda}};$$

$$\hat{L}_{\lambda\lambda\alpha}=\frac{d^3L\left(\lambda,\alpha\right)}{d\hat{\lambda}.d\hat{\lambda}.d\hat{\alpha}};\hat{L}_{\alpha\alpha\alpha}=\frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}.d\hat{\alpha}.d\hat{\alpha}};\hat{L}_{\alpha\lambda\alpha}=\frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}\,d\hat{\lambda}.d\hat{\alpha}};$$

$$;\hat{P}_{\alpha}=\frac{dG\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}};\hat{P}_{\lambda}=\frac{dG\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}};$$

2.2.1 Bayes Estimator of λ

The Bayes estimator of λ under General Entropy loss function is given as

$$\hat{\lambda}_{GE}=\left[E_{\lambda}\left(\lambda^{-c_1}\right)\right]^{-\frac{1}{c_1}} \tag{14}$$

where E_{λ} stands for posterior expectations and is defined below

$$E_{\lambda}\left(\lambda^{-c_1}\right)=\int_{(\lambda,\alpha)}\lambda^{-c_1}p\left(\lambda,\alpha\right)d\left(\lambda,\alpha\right)$$

$$=\frac{\int_{(\lambda,\alpha)}u\left(\lambda,\alpha\right)e^{L(\lambda,\alpha)+G(\lambda,\alpha)}d\left(\lambda,\alpha\right)}{\int_{(\lambda,\alpha)}e^{L(\lambda,\alpha)+G(\lambda,\alpha)}d\left(\lambda,\alpha\right)}$$

Here,

$$u\left(\hat{\lambda},\hat{\alpha}\right)=\lambda^{-c_1}$$

$$L\left(\hat{\lambda},\hat{\alpha}\right) = n\log\alpha + n\log\lambda + (\alpha-1)\left[\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)\right] - \lambda\sum_{i=1}^{n}x_i$$

$$P\left(\hat{\lambda},\hat{\alpha}\right) = c\log m - \log\alpha - \log\Gamma c + (c-1)\log\lambda - m\lambda$$

Evaluating u-terms, L-terms, and p-terms mentioned above at point ($\hat{\lambda},\hat{\alpha}$) and using (13) we get

$$E_\lambda\left[\lambda^{-c_1}\right]=\lambda^{-c_1}+\frac{1}{2}\left[\left(\frac{c_1(c_1+1)\lambda^{-(c_1+2)}-2c_1\lambda^{-(c_1+1)}\left(\frac{c-1}{\lambda}-m\right)}{\left(\frac{n}{\lambda^2}+(\alpha-1)\left[\sum_{i=1}^{n}\frac{x_i{}^2e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]\right)}\right)\right.$$

$$\left.-\left(\frac{c_1\lambda^{-(c_1+1)}\left(\frac{2n}{\lambda^3}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i{}^3e^{-\lambda x_i}\left(1+e^{-\lambda x_i}\right)}{\left(1-e^{-\lambda x_i}\right)^3}\right)}{\left(\frac{n}{\lambda^2}+(\alpha-1)\left[\sum_{i=1}^{n}\frac{x_i{}^2e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]\right)^2}\right)\right]$$

Thus, Bayes estimator of λ under General Entropy loss function

$$\hat{\lambda}_{GE}=\left[E_\lambda\left(\lambda^{-c_1}\right)\right]^{-\frac{1}{c_1}}$$

$$=\lambda^{-c_1}+\frac{1}{2}\left[\left(\frac{c_1(c_1+1)\lambda^{-(c_1+2)}-2c_1\lambda^{-(c_1+1)}\left(\frac{c-1}{\lambda}-m\right)}{\left(\frac{n}{\lambda^2}+(\alpha-1)\left[\sum_{i=1}^{n}\frac{x_i{}^2e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]\right)}\right)\right.$$

$$-\left(\frac{c_1\lambda^{-(c_1+1)}\left(\frac{2n}{\lambda^3}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\,3}e^{-\lambda x_i}\left(1+e^{-\lambda x_i}\right)}{\left(1-e^{-\lambda x_i}\right)^3}\right)}{\left(\frac{n}{\lambda^2}+(\alpha-1)\left[\sum_{i=1}^{n}\frac{x_i^{\,2}e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]\right)^2}\right)\Bigg]\Bigg]$$

It may be noted that as n $\to\infty$, $\hat{\lambda}_{GE}\to\hat{\lambda}$ i.e.

when sample size n is sufficiently large, the Bayes estimator of Generalized-Exponential parameter λ under General Entropy loss function tends to maximum likelihood estimator of λ .

2.2.2 Bayes Estimator of α

The Bayes estimator of α under General Entropy loss function is

$$\hat{\alpha}_{GE}=\left[E_\alpha\left(\alpha^{-c_1}\right)\right]^{-\frac{1}{c_1}}$$

where E_α stands for posterior expectations and is defined below:

$$E_\alpha\left(\alpha^{-c_1}\right)=\int_{(\lambda,\alpha)}\alpha^{-c_1}p(\lambda,\alpha)d(\lambda,\alpha)$$

$$=\frac{\int_{\hat{\lambda},\hat{\alpha}}u\left(\hat{\lambda},\hat{\alpha}\right)e^{L\left(\hat{\lambda},\hat{\alpha}\right)+p\left(\hat{\lambda},\hat{\alpha}\right)}d(\lambda,\alpha)}{\int_{\hat{\lambda},\hat{\alpha}}e^{L\left(\hat{\lambda},\hat{\alpha}\right)+p\left(\hat{\lambda},\hat{\alpha}\right)}d(\lambda,\alpha)}$$

Here,

$$u\left(\hat{\lambda},\hat{\alpha}\right)=\alpha^{-c_1},$$

$$L\left(\hat{\lambda},\hat{\alpha}\right)\ =n\log\alpha+n\log\lambda+(\alpha-1)\left[\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)\right]-\lambda\sum_{i=1}^{n}x_i$$

$$P\left(\hat{\lambda},\hat{\alpha}\right)\ =\ c\log m-\log\alpha-\log\Gamma c+(c-1)\log\lambda-m\lambda$$

Evaluating u-terms, L-terms, and p-terms using (13) we get

$$E_\alpha\left[\alpha^{-c_1}\right] = \alpha^{-c_1}\left\langle 1+\frac{1}{2}c_1\frac{\alpha^2}{n}\left[\left\{(c_1+3)\alpha^{-2}\right\}\right.\right.$$

$$\left.\left.-\alpha^{-1}\left(\frac{2}{\alpha}-\frac{\sum_{i=1}^{n}\frac{x_i^2 e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}}{\frac{n}{\lambda^2}+(\alpha-1)\left[\sum_{i=1}^{n}\frac{x_i^2 e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]}\right)\right]\right\rangle$$

Thus, the Bayes estimator of α under General Entropy loss function is given by

$$\hat{\alpha}_{GE} = \left[E_\alpha\left(\alpha^{-c_1}\right)\right]^{-\frac{1}{c_1}}$$

$$\hat{\hat{\alpha}}_{GE} = \left[\hat{\alpha}^{-c_1}\left\langle 1+\frac{1}{2}c_1\frac{\hat{\alpha}^2}{n}\left[\left\{(c_1+3)\hat{\alpha}^{-2}\right\}\right.\right.\right.$$

$$\left.\left.\left.-\hat{\alpha}^{-1}\left(\frac{2}{\hat{\alpha}}-\frac{\sum_{i=1}^{n}\frac{x_i^2 e^{-\hat{\lambda} x_i}}{\left(1-e^{-\hat{\lambda} x_i}\right)^2}}{\frac{n}{\hat{\lambda}^2}+(\hat{\alpha}-1)\left[\sum_{i=1}^{n}\frac{x_i^2 e^{-\hat{\lambda} x_i}}{\left(1-e^{-\hat{\lambda} x_i}\right)^2}\right]}\right)\right]\right\rangle\right]^{-\frac{1}{c_1}}$$

It may be noted that as $n \to \infty$, $\hat{\alpha}_{GE} \to \hat{\alpha}$ i.e.

when sample size n is sufficiently large then Bayes estimator of Generalized-Exponential parameter α under General Entropy loss function tends to maximum likelihood estimator of α .

3. Comparison

In the previous section we have seen that the Lindley's approximation provides the Bayes estimators in a nice closed form as function of maximum likelihood estimators. But, the maximum likelihood estimators are not expressible in closed form. Therefore, analytical expressions for the risk of the proposed estimator as well as maximum likelihood estimator cannot be obtained. Under such a situation comparison of the proposed estimator with maximum likelihood estimator based on their risk can only be studied using simulation technique. It may be noted that the risk of the estimators will be function of sample size, population parameters, parameters of the prior distribution (hyper-parameters) and loss function parameters. We have obtained the simulated risks for sample size n = 10(10)40. The different values of parameters of the distribution considered by us are loss parameter c_1 =0.5(0.5)2.0; scale parameter of model λ = 1.5(0.5)3.0; shape parameter of model α = 1.5(0.5)3.0; and prior parameters m = 1(1)5 and c = 1(1)5. For all the combination of the above-mentioned values of sample size and other parameters, risks of the estimators have been estimated on the basis of 2000 simulated samples. After an extensive study of the results, thus obtained, conclusions were drawn regarding the behaviour of the estimator which is summarized below. However, only a few results are shown in the form of graphs due to paucity of space.

While studying the effect of variation in the value of scale parameter (m) and shape parameter (c) of prior distribution, it is noted that risk of the proposed Bayes estimator of λ increases as (c) increases and decreases as (m) increases, although the variation is very little. Needless to mention that risks of maximum likelihood estimator of λ and α and proposed Bayes estimator of α are constant as there is no effect of (c) and (m) on these. Keeping this in mind, only for c=1 and m=1, the graphs of risks are included in this paper. It may be noted here that the trend is almost same as discussed above for positive and negative value of c_1 . It may also be worthwhile to mention that the risks of proposed Bayes estimators are always found to be less then those of maximum likelihood estimator.

Let us consider the effect of sample size n on the risk of the estimators, when loss parameter, prior parameter and model parameter are fixed at any particular value. For positive value of c_1 i.e. when overestimation is more serious than underestimation. It was noted from simulated risks that as sample size increases, the risks of estimator of λ as well as the risks of the estimator of α decreases, which is an obvious result. Further, it may be seen that in most of the cases, the risk of Bayes estimator of α is less than that of the risk of the maximum likelihood estimator. It may be also noted that as sample size increases the difference between the risk of Bayes estimator and maximum likelihood estimator decreases. For negative value of c_1, i.e. when underestimation is more serious than overestimation, then also, risks of estimator of λ and α decrease with increase in sample size. The risks of proposed Bayes estimator of λ and α are always less than the risk of maximum likelihood estimator of λ and α. (See figure 1, 2, 3, 4)

Now we observe the effect of **scale parameter** λ on the risks of estimator of λ keeping values of other parameters fixed for positive value of c_1. It may be noted that as λ increases, risks of estimator of λ first decreases and then increases. The risk of the maximum likelihood estimator first decreases, then increases and again increases for further increases in λ but variation is less in magnitude than that of Bayes estimator. For negative value of c_1, the risks of estimator of λ first decreases and then increases for increase in the value of λ. The maximum likelihood estimator has similar trend. It is also noticed that risks of proposed Bayes estimator of λ is always less than that of Maximum Likelihood estimator of λ which shows that proposed Bayes estimator performs well in whole of the considered values of the parameters (see figure 5, 6).

Studying the variation of **shape parameter** α on the risks of the estimator of α, it is noted that for positive value of c_1, as α increases the risk of the proposed Bayes estimator as well as maximum likelihood estimator increases. However, the risk of Bayes estimator is always less then that of maximum likelihood estimator. For negative value of c_1, i.e. when underestimation is more serious than overestimation, the risks of Bayes estimator as well as maximum likelihood estimator of α increases as value of α increases, then decreases and again increases for further increases in the value α. However, the magnitude of risk of Bayes estimator is almost equal or slightly less then that of maximum likelihood estimator (see figure 7, 8).

From the results, it is observed that for small value of c_1 the risks of Bayes estimator is increasing slightly less than the risk of the maximum likelihood estimator. However, as the magnitude of loss parameter c_1 increases, risks of the estimators of λ and α increases. It is also noticed that increases in the risks of maximum likelihood estimator are more significant than those of Bayes estimator, giving a greater gain due to use of Bayes estimator for large value of $|c_1|$. It may also be noted here that difference in risk of Bayes and maximum likelihood estimator is more for positive value of c_1 then that of negative values of c_1 (see figure 9, 10).

4. Conclusion

The behaviour of the proposed Bayes estimator under General Entropy loss in comparison to maximum likelihood estimator has been studied in above section. It is noted that for positive c_1, i.e. when overestimation is more serious than underestimation, proposed Bayes estimator of λ performs better than maximum likelihood estimator of λ for large portion of parametric space and proposed Bayes estimator of α performs better than maximum likelihood estimator of α for whole parametric space. For negative c_1, i.e. when underestimation is more serious than overestimation, proposed Bayes estimator of λ and α performs better than maximum likelihood estimator of λ and α respectively for whole parametric space. Thus, the proposed Bayes estimator may be recommended for its use.

Figure 1. Risk of estimator of λ when c_1 is positive as function of sample size N

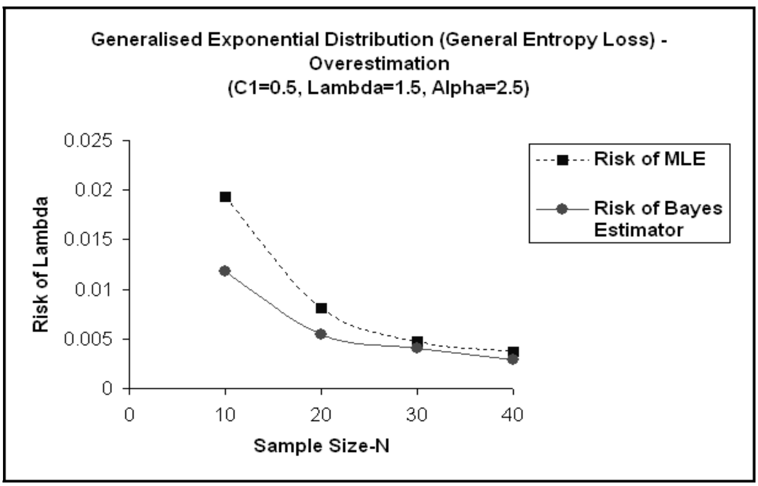


Figure 2. Risk of estimator of λ when c_1 is negative as function of sample size N

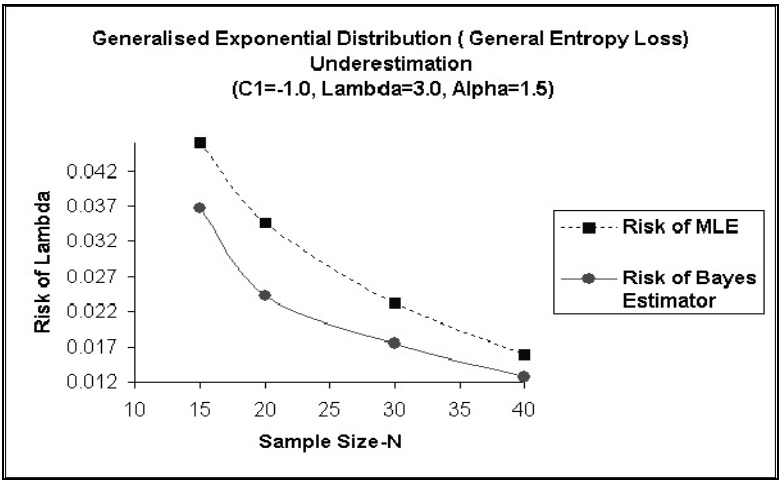


Figure 3. Risk of estimator of α when c_1 is positive as function of sample size of N

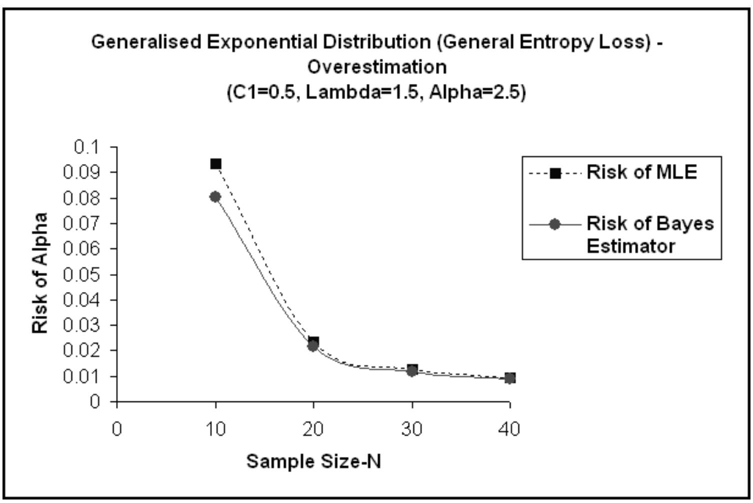


Figure 4. Risk of estimator of α when c_1 negative as function of sample size of N

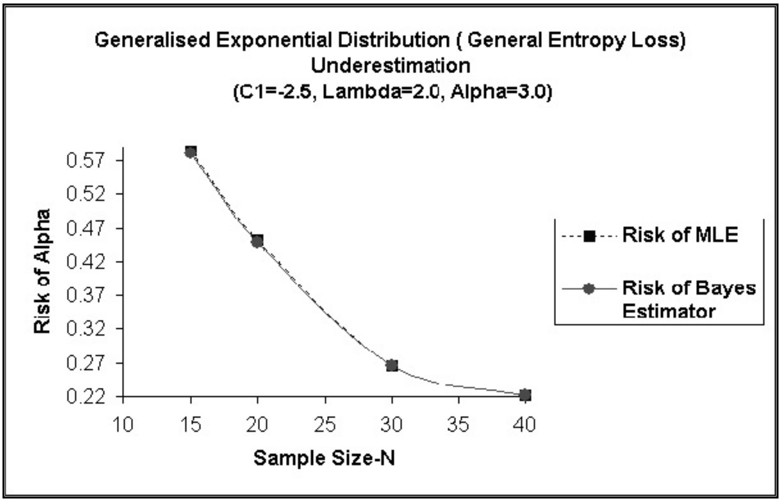


Figure 5. Risk of estimator of λ when c_1 is positive as function λ

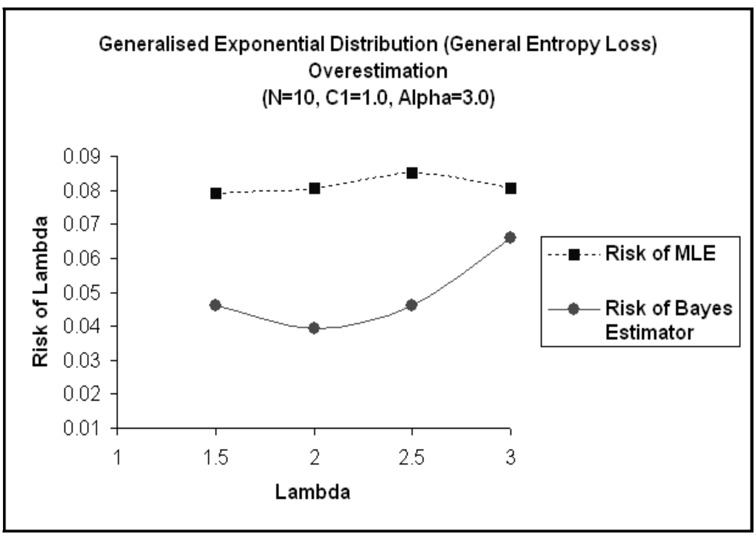


Figure 6. Risk of estimator of λ when c_1 negative as function λ

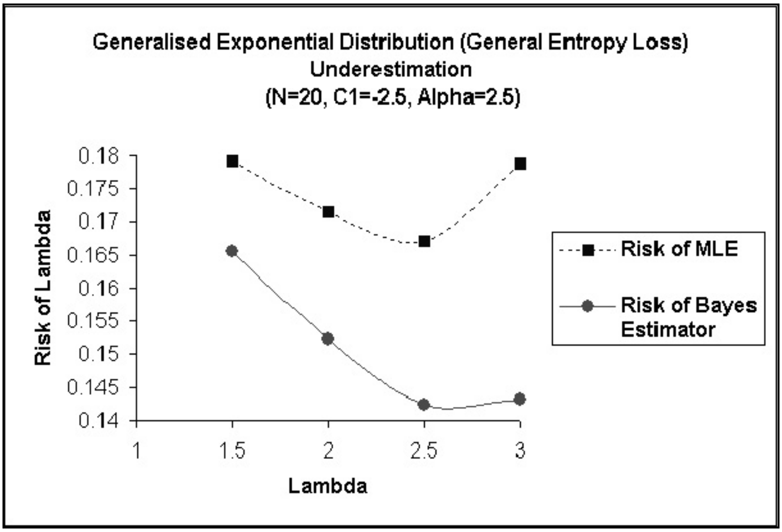


Figure 7. Risk of estimator of α when c_1 is positive as function α

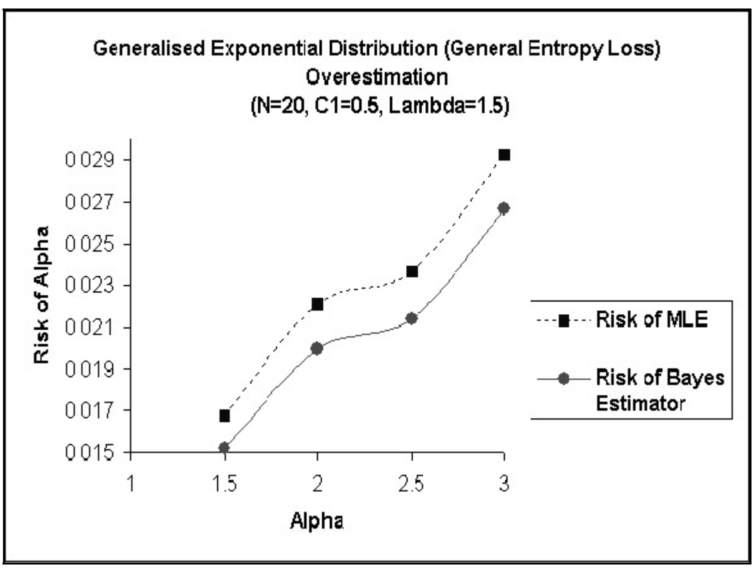


Figure 8. Risk of estimator of α when c_1 negative as function α

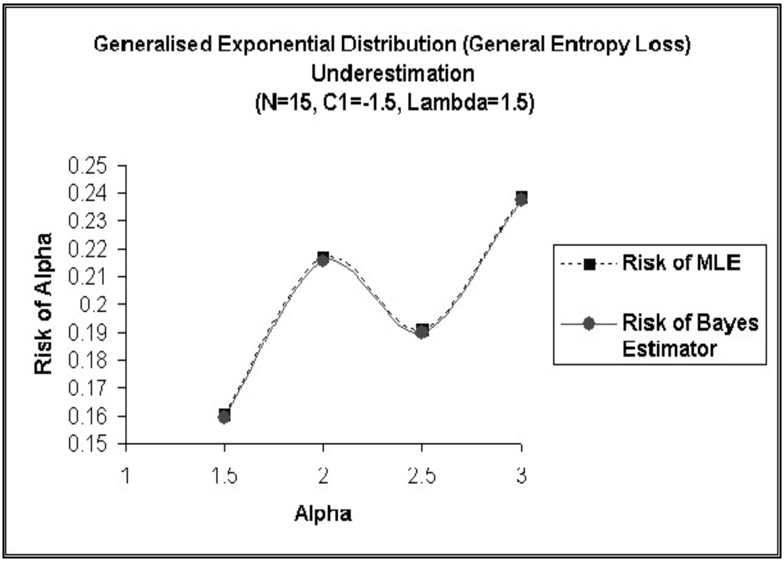


Figure 9. Risk of estimator of λ as function c_1

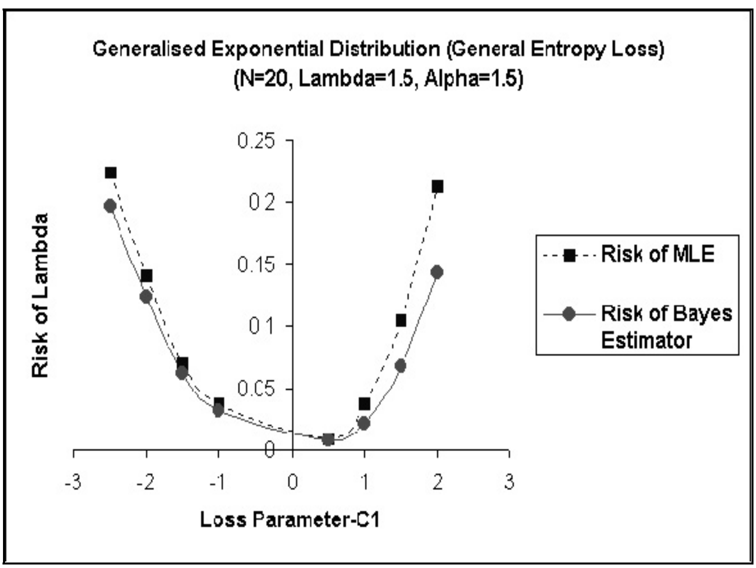
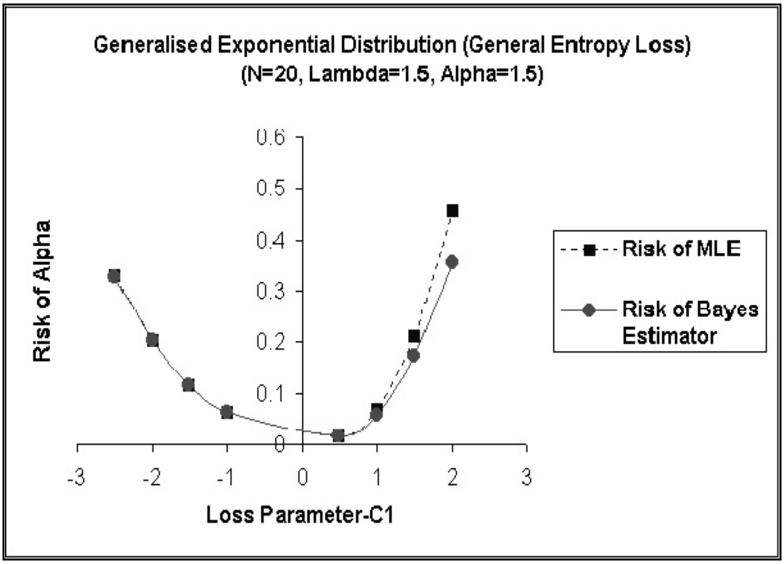


Figure 10. Risk of estimator of α as function c_1



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